

THE PREPROJECTIVE ALGEBRA OF A MODULATED GRAPH

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The present paper generalizes a recent result of I.M. Gelfand and V.A. Ponomarev [4] reported at the Conference by V.A. Rojter.

A modulated graph $\mathcal{M} = (F_i, {}_iM_j, \epsilon_i^j)_{i,j \in I}$ is given by division rings F_i for all $i \in I$, by bimodules ${}_i({}_iM_j)_{F_j}$ for all $i \neq j$ in I finitely generated on both sides and by non-degenerate bilinear forms $\epsilon_i^j: {}_iM_j \otimes {}_jM_i \rightarrow F_i$; here, I is a finite index set. Note that the forms ϵ_i^j give rise to canonical elements $c_j^i \in {}_jM_i \otimes {}_iM_j$. Namely, if x_1, \dots, x_d is a basis of $({}_jM_i)_{F_i}$ and y_1, \dots, y_d the corresponding dual basis of ${}_i({}_iM_j)_{F_j}$ with respect to ϵ_i^j , then $c_j^i = \sum_p x_p \otimes y_p$; see section 1.

Define the ring $\Pi(\mathcal{M})$ as follows. Let $T(\mathcal{M})$ be the tensor ring of \mathcal{M} : $T(\mathcal{M}) = \bigoplus_{t \in \mathbb{N}} T_t$, where $T_0 = \prod_i F_i$, $T_1 = \bigoplus_{i,j} {}_iM_j$ and $T_{t+1} = T_1 \otimes_{T_0} T_t$ with the multiplication given by the tensor product. Then, by definition, $\Pi(\mathcal{M}) = T(\mathcal{M}) / \langle c \rangle$, where $\langle c \rangle$ is the principal ideal of $T(\mathcal{M})$ generated by the element $c = \sum_{i,j} c_j^i$.

Let Ω be an (admissible) orientation of \mathcal{M} ; thus, for every pair i, j with ${}_iM_j \neq 0$, we prescribe an order indicated by an arrow $i \rightarrow j$, or $i \leftarrow j$ in such a way that no oriented cycles occur.

Let $R(\mathcal{M}, \Omega)$ be the corresponding tensor ring of (\mathcal{M}, Ω) : $R(\mathcal{M}, \Omega) =$

$$\bigoplus_{t \in \mathbb{N}} R_t \text{ with } R_0 = \prod_i F_i, \quad R_1 = \bigoplus_{i \rightarrow j} {}_iM_j \text{ and } R_{t+1} = R_1 \otimes_{R_0} R_t.$$

For the representation theory of $R(\mathcal{M}, \Omega)$ we refer to [3].

THEOREM. For each orientation Ω of \mathcal{M} , $R(\mathcal{M}, \Omega)$ is a subring of $\prod \mathcal{M}$ and, as a (right) $R(\mathcal{M}, \Omega)$ -module, $\prod \mathcal{M}$ is the direct sum of all indecomposable preprojective $R(\mathcal{M}, \Omega)$ -modules (each occurring with multiplicity one).

This theorem suggests to call $\prod \mathcal{M}$ the preprojective algebra of \mathcal{M} . Recall that an indecomposable $R(\mathcal{M}, \Omega)$ -module P is preprojective if and only if there is only a finite number of indecomposable modules X with $\text{Hom}(X, P) \neq 0$.

COROLLARY. The ring $\prod \mathcal{M}$ is artinian if and only if the modulated graph is a disjoint union of Dynkin graphs.

Observe that if \mathcal{M} is a K -modulation (where K is a commutative field), then $\prod \mathcal{M}$ is a K -algebra. In this case, the corollary may be reformulated as follows: The algebra $\prod(\mathcal{M})$ is finite-dimensional if and only if \mathcal{M} is a disjoint union of Dynkin graphs.

Consider, in particular, the case when (\mathcal{M}, Ω) is given by a quiver; thus, $F_i = K$ for all i and ${}_iM_j$ is a direct sum of a finite number of copies of ${}_K^K$. For every arrow x of the quiver, define an "inverse" arrow x^* whose end is the origin of x and whose origin is the end of x . Then $T(\mathcal{M})$ is the path algebra generated by all arrows x and x^* , and $\prod(\mathcal{M})$ is the quotient of $T(\mathcal{M})$ by the ideal generated by the element $\sum_{\text{all } x} (xx^* + x^*x)$.

COROLLARY. If (\mathcal{M}, Ω) is given by a quiver, then $\prod(\mathcal{M})$ is finite-dimensional if and only if the quiver is of finite type.

For a quiver which is a tree, the last result has been announced by A.V. Rojter [6] in his report on the paper [4]. In contrast to the proofs in [4], our approach avoids use of reflection functors and is based on the explicit description of the category

$P(\mathcal{M}, \Omega)$ of all preprojective $R(\mathcal{M}, \Omega)$ -modules. The authors are indebted to P. Gabriel for pointing out that the theorem is, in the case when (\mathcal{M}, Ω) is given by a quiver, also due to Ch. Riedtmann [7].

1. Preliminaries on dualization

Given a finite-dimensional vector space ${}_F M$, denote by ${}^* M$ its (left) dual space $\text{Hom}({}_F M, {}_F F)$. If ${}_F M_G$ is a bimodule and ${}_G X, {}_F Y$ vector spaces, the adjoint map $\bar{f}: X \rightarrow {}^* M \otimes_F Y$ to a map $f: M \otimes_G X \rightarrow Y$ is given by $\bar{f}(x) = \sum_{p=1}^d \phi_p \otimes f(m_p \otimes x)$, where $x \in X$, $\{m_1, m_2, \dots, m_d\}$ is a basis of ${}_F M$ and $\{\phi_1, \phi_2, \dots, \phi_d\}$ is the respective dual basis of $({}^* M)_F$. In particular, if M is an $\text{End } Y - \text{End } X$ -submodule of the bimodule $\text{Hom}(X, Y)$ and $\chi_M: M \otimes X \rightarrow Y$ the evaluation map $\chi_M(m \otimes x) = m(x)$, then $\bar{\chi}_M(x) = \sum_p \phi_p \otimes m_p(x)$. Note that $\bar{\chi}_M$ is a (left) G -homomorphism.

Now, given bimodules ${}_F M_G, {}_G N_F$ such that ${}_F M$ and N_F are finite dimensional, let $\varepsilon: M \otimes_G N \rightarrow F$ be a non-degenerate bilinear form. Thus, the adjoint $\bar{\varepsilon}$ is an isomorphism $\bar{\varepsilon}: N \rightarrow {}^* M$; let $\{n_1, n_2, \dots, n_d\}$ be a basis of N_F and $\{\phi_1, \phi_2, \dots, \phi_d\}$ the basis of $({}^* M)_F$ such that $\phi_p = \bar{\varepsilon}(n_p)$ for all $1 \leq p \leq d$. Furthermore, let $\{m_1, m_2, \dots, m_d\}$ be the dual basis of ${}_F M$. Thus,

$$\varepsilon(m_p \otimes n_q) = (m_p)[\bar{\varepsilon}(n_q)] = (m_p) \phi_q = \delta_{pq}.$$

Define the canonical element c_ε of $N \otimes_F M$ (with respect to ε) by

$$c_\varepsilon = \sum_{p=1}^d n_p \otimes m_p.$$

Lemma 1.1. *The element c_ε does not depend on the choice of a basis.*

Proof. Let $\{n'_1, n'_2, \dots, n'_d\}$ and $\{m'_1, m'_2, \dots, m'_d\}$ be another bases of N_F and ${}_F M$, respectively, so that

$$\varepsilon(m'_p \otimes n'_q) = \delta_{pq}.$$

Then $n'_q = \sum_j n_j b_{jq}$ and $m'_p = \sum_i a_{pi} m_i$ with b_{jq} and a_{pi} from F .

Since $\delta_{pq} = \varepsilon(m'_p \otimes n'_q) = \sum_{i,j} a_{pi} \varepsilon(m_i \otimes n_j) b_{jq} = \sum_i a_{pi} b_{iq}$,

we have also $\sum_p b_{jp} a_{pi} = \delta_{ji}$.

Thus,

$$\begin{aligned} \sum_p n'_p \otimes m'_p &= \sum_{i,j,p} n_j b_{jp} \otimes a_{pi} m_i \\ &= \sum_{i,j} n_j \left(\sum_p b_{jp} a_{pi} \right) \otimes m_i = \sum_i n_i \otimes m_i . \end{aligned}$$

If we take, in particular, ${}_G N_F = {}^*({}_F M_G)$ and the evaluation map $\chi : M \otimes_G N \rightarrow F$ defined by

$$\chi(m \otimes \phi) = (m)\phi ,$$

we obtain, for every bimodule M , the canonical element $c(M) = c_\chi$.

Given a bimodule ${}_F M_G$, define the higher dual spaces ${}^{(t)}_F M_G$ inductively by

$${}^{(t+1)}_F M_G = {}^*({}^{(t)}_F M_G) .$$

Thus, ${}^{(t)}_M$ is an F - G -bimodule for t even and a G - F -bimodule for t odd.

Lemma 1.2. Let ${}_F M_G$ and ${}_G N_F$ be bimodules and $\varepsilon : M \otimes_F N \rightarrow F$ and $\delta : {}_G N_F \otimes_G M_G \rightarrow G$ non-degenerate bilinear forms. Define the maps t_η inductively as follows:

$$\begin{aligned} 0_\eta &= 1_M : {}_F M_G \rightarrow {}^{(0)}_M = M ; \\ 1_\eta &= \bar{\varepsilon} : {}_G N_F \rightarrow {}^{(1)}_M = {}^*_M ; \\ 2r_\eta &= \overline{\delta[({}^{(2r-1)}_\eta)^{-1} \otimes 1_M]} : {}_F M_G \rightarrow {}^{(2r)}_M \text{ and} \\ 2r+1_\eta &= \overline{\varepsilon[({}^{(2r)}_\eta)^{-1} \otimes 1_N]} : {}_G N_F \rightarrow {}^{(2r+1)}_M . \end{aligned}$$

Then

$$[{}^{2r+1}_\eta \otimes {}^{2r+2}_\eta] (c_\varepsilon) = c({}^{(2r)}_M) \text{ and } [{}^{2r}_\eta \otimes {}^{2r+1}_\eta] (c_\delta) = c({}^{(2r+1)}_M) .$$

Proof. Recall that $c_\varepsilon = \sum_p n_p \otimes m_p$, where $\{m_1, m_2, \dots, m_d\}$ is a basis of ${}_F M$ and $\{n_1, n_2, \dots, n_d\}$ the dual basis of ${}_F N$ with respect to ε . Hence, in order to prove the first equality, it is sufficient to show that, for $m \in M$ and $n \in N$,

$$\delta(n \otimes m) = ({}^{2r+1}_\eta(n)) [{}^{2r+2}_\eta(m)] .$$

$$\begin{aligned}
 \text{But, } \{^{2r+1}_{\eta(n)}\} [^{2r+2}_{\eta(m)}] &= \{^{2r+1}_{\eta(n)}\} \overline{\delta[\{^{2r+1}_{\eta}\}^{-1} \otimes 1_M]} [^{(m)}] = \\
 &= \delta[\{^{2r+1}_{\eta}\}^{-1} \otimes 1_M] (^{2r+1}_{\eta(n)}) = \delta[\{^{2r+1}_{\eta}\}^{-1} \{^{2r+1}_{\eta(n)} \otimes m\}] = \\
 &= \delta(n \otimes m) .
 \end{aligned}$$

Similarly, since

$$\begin{aligned}
 \{^{2r}_{\eta(m)}\} [^{2r+1}_{\eta(n)}] &= \{^{2r}_{\eta(m)}\} \overline{\varepsilon[\{^{2r}_{\eta}\}^{-1} \otimes 1_N]} [^{(n)}] = \\
 &= \varepsilon[\{^{2r}_{\eta}\}^{-1} \otimes 1_N] (^{2r}_{\eta(m)}) = \varepsilon[\{^{2r}_{\eta}\}^{-1} \{^{2r}_{\eta(m)} \otimes n\}] = \\
 &= \varepsilon(m \otimes n) ,
 \end{aligned}$$

we can derive the second equality for $c(^{(2r+1)}_M)$.

2. Irreducible maps

Recall the definition of an irreducible map [2]: a map $f : X \rightarrow Y$ is called irreducible if f is neither a split monomorphism nor a split epimorphism and if, for every factorization $f = f'f''$, either f'' is a split monomorphism or f' is a split epimorphism. Also, recall the definition of the radical of a module category. If X and Y are indecomposable modules, let $\text{rad}(X, Y)$ be the set of all non-invertible homomorphisms. If $X = \bigoplus_p X_p$ and $Y = \bigoplus_q Y_q$ with indecomposable modules X_p and Y_q , define $\text{rad}(X, Y) = \bigoplus_{p,q} \text{rad}(X_p, Y_q)$, using the identification $\text{Hom}(X, Y) = \bigoplus_{p,q} \text{Hom}(X_p, Y_q)$. The square $\text{rad}^2(X, Y)$ of the radical is thus the set of all homomorphisms $f : X \rightarrow Y$ such that $f = f'f''$, where $f'' \in \text{rad}(X, Z)$ and $f' \in \text{rad}(Z, Y)$ for some module Z . Note that both rad and rad^2 are ideals in our module category; in particular, $\text{rad}(X, Y)$ and $\text{rad}^2(X, Y)$ are $\text{End } Y$ - $\text{End } X$ - submodules of the bimodule $\text{End } Y^{\text{Hom}(X, Y)}_{\text{End } X}$. For indecomposable X and Y , the elements in $\text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$ are just the irreducible maps. In this case, we write $\text{Irr}(X, Y) = \text{rad}(X, Y) / \text{rad}^2(X, Y)$, and call $\text{Irr}(X, Y)$ the *bimodule of irreducible maps* (see [5]). In what follows, our main objective is to select a direct complement of $\text{rad}^2(X, Y)$ in $\text{rad}(X, Y)$ which is an $\text{End } Y$ - $\text{End } X$ - submodule, and realize in this way

$\text{Irr}(X, Y)$ as a subset of $\text{Hom}(X, Y)$ rather than just as a factor group. We shall select such complements inductively, using Auslander-Reiten sequences.

Recall that an exact sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is called an Auslander-Reiten sequence if both maps f and g are irreducible. This implies that both modules X and Z are indecomposable, X is not injective and Z is not projective. Conversely, given an indecomposable non-injective module X , there exists an Auslander-Reiten sequence starting with X , and also dually, given an indecomposable non-projective Z , there is an Auslander-Reiten sequence ending with Z . Moreover, if $0 \rightarrow X \xrightarrow{f} Y \rightarrow Z \rightarrow 0$ is an Auslander-Reiten sequence and $h : X \rightarrow X'$ is a map which is not a split monomorphism, then there exists $\alpha : Y \rightarrow X'$ such that $h = \alpha f$. (For all these properties, we refer to [2]).

In the sequel, we will consider direct sums of the form $\bigoplus_Y U(Y)$, where $U(Y)$ is an abelian group depending on Y , with Y ranging over "all" indecomposable modules. Here, of course, we choose first fixed representatives Y of all isomorphism classes of indecomposable modules and then index the direct sum by these representatives. In fact, all direct sum which will occur in this way will have even only a finite number of non-zero summands.

PROPOSITION 2.1. *Let X be an indecomposable non-injective module and G be a division ring with*

$$\text{End } X = G \oplus \text{rad } \text{End } X.$$

Assume that, for every indecomposable module Y , there is given a direct complement $M(X, Y)$ of $\text{rad}^2(X, Y)$ in $\text{End } Y \text{ rad}(X, Y)_G$. Let

$$0 \longrightarrow X \xrightarrow{\begin{pmatrix} \overline{X} \\ M(X, Y) \end{pmatrix} Y} \bigoplus_Y {}^*M(X, Y) \otimes_Y Y \xrightarrow{\pi} Z \longrightarrow 0$$

*be exact. Then, this is an Auslander-Reiten sequence. Moreover, G embeds into the endomorphism ring $\text{End } Z$ of Z as a radical complement, and for every Y , there is an embedding σ of ${}^*M(X, Y)$ onto a complement of $\text{rad}^2(Y, Z)$ in $\text{rad}(Y, Z)_{\text{End } Y}$ such that*

$$\chi_{\mathcal{O}}^* M(X, Y) = \pi \mid {}^* M(X, Y) \otimes Y.$$

Proof. Let

$$0 \longrightarrow X \xrightarrow{(f'_{Y,p})_{Y,p}} \bigoplus_{Y,p=1}^{d_Y} Y \longrightarrow Z' \longrightarrow 0$$

be an Auslander-Reiten sequence starting with X , where $f'_{Y,p} : X \rightarrow Y$ for $1 \leq p \leq d_Y$. Then the residue classes of the elements $f'_{Y,1}, f'_{Y,2}, \dots, f'_{Y,d_Y}$ form a basis of the G -vector space $\text{rad}(X, Y)_G / \text{rad}^2(X, Y)_G$ (see Lemma 2.5 of [5]). Let $f_{Y,1}, f_{Y,2}, \dots, f_{Y,d_Y}$ be a G -basis of $M(X, Y)$. By the factorization property of Auslander-Reiten sequences, there is a map

$$\alpha : \bigoplus_{Y,p=1}^{d_Y} Y \longrightarrow \bigoplus_{Y,p=1}^{d_Y} Y$$

such that $\alpha \circ (f'_{Y,p})_{Y,p} = (f_{Y,p})_{Y,p}$. It follows that α is an automorphism. For, let $E = \text{End}(\bigoplus_{Y,p=1}^{d_Y} Y)$ and consider the residue class $\bar{\alpha}$ of α in $E / \text{rad } E$. Also, consider the factor group

$$M = \text{rad}(X, \bigoplus_{Y,p=1}^{d_Y} Y) / \text{rad}^2(X, \bigoplus_{Y,p=1}^{d_Y} Y),$$

and let \bar{f} and \bar{f}' be the residue classes of $f = (f_{Y,p})_{Y,p}$ and $f' = (f'_{Y,p})_{Y,p}$, respectively. Then $\text{rad } E$ annihilates M , and the equality $\bar{\alpha} \bar{f}' = \bar{f}$ shows that $\bar{\alpha}$ induces base changes between the bases $(\bar{f}_{Y,p})_p$ and $(\bar{f}'_{Y,p})_p$ of $\text{Irr}(X, Y)$. This implies that $\bar{\alpha}$ is invertible. Since $\text{rad } E$ is nilpotent, α is invertible, as well. Thus, we can form the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f'} & \bigoplus_{Y,p=1}^{d_Y} Y & \longrightarrow & Z' \longrightarrow 0 \\ & & \parallel & & \downarrow \alpha & & \downarrow \beta \\ 0 & \longrightarrow & X & \xrightarrow{f} & \bigoplus_{Y,p=1}^{d_Y} Y & \xrightarrow{\pi} & Z \longrightarrow 0 \end{array},$$

where both α and β are isomorphisms. As a consequence, also the lower sequence is an Auslander-Reiten sequence.

Note that we can rewrite $\bigoplus_{p=1}^{d_Y} Y$ as ${}^*M(X,Y) \otimes_{\text{End } Y} Y$, and then $(f_{Y,p})_p$ becomes $\bar{\chi}_{M(X,Y)}$. For, if $\phi_{Y,1}, \phi_{Y,2}, \dots, \phi_{Y,d_Y}$ is the dual basis of ${}^*M(X,Y)_{\text{End } Y}/\text{rad End } Y$ with respect to the basis $f_{Y,1}, f_{Y,2}, \dots, f_{Y,d_Y}$ of $\text{End } Y/\text{rad End } Y^{M(X,Y)}$, then we identify

$${}^*M(X,Y) \otimes_{\text{End } Y} Y = \bigoplus_{p=1}^{d_Y} \phi_{Y,p} \otimes Y \approx \bigoplus_{p=1}^{d_Y} Y,$$

and

$$\bar{\chi}_{M(X,Y)}(x) = \sum_{p=1}^{d_Y} \phi_{Y,p} \otimes f_{Y,p}(x)$$

is identified with $(f_{Y,p}(x))_p$.

Now, ${}^*M(X,Y)$ is a left G -module, and

$$\bar{\chi}_{M(X,Y)} : X \longrightarrow {}^*M(X,Y) \otimes_{\text{End } Y} Y$$

is a G -module homomorphism. Hence, under $(\bar{\chi}_{M(X,Y)})_Y$, the module X becomes a G -submodule of $\bigoplus_Y {}^*M(X,Y) \otimes_{\text{End } Y} Y$, and therefore also the factor module Z has a left G -module structure. Thus, G embeds canonically into $\text{End } Z$ and in this way, G becomes a radical complement. This follows from the canonical isomorphism

$$\text{End } X/\text{rad End } X \approx \text{End } Z/\text{rad End } Z,$$

which is always valid for the outer terms of an Auslander-Reiten sequence.

The restriction of π to ${}^*M(X,Y) \otimes Y$ defines a map σ of ${}^*M(X,Y)$ into $\text{Hom}(Y,Z)$ which is a G - $\text{End } Y$ -homomorphism. If we denote again by $\phi_{Y,1}, \phi_{Y,2}, \dots, \phi_{Y,d_Y}$ an $\text{End } Y/\text{rad End } Y$ -basis of ${}^*M(X,Y)$, then $\pi|_{{}^*M(X,Y) \otimes_{\text{End } Y} Y} \longrightarrow Z$ can be identified with

$$(\phi_{Y,p})_p : \bigoplus_{p=1}^{d_Y} Y \approx \bigoplus_{p=1}^{d_Y} \phi_{Y,p} \otimes Y \longrightarrow Z.$$

Again, using Lemma 2.5 of [5], we see that the residue classes of $\phi_{Y,1}, \phi_{Y,2}, \dots, \phi_{Y,d_Y}$ in $\text{Irr}(Y, Z)$ form an $\text{End } Y/\text{rad End } Y$ -basis and that ${}^*M(X, Y)$ is therefore mapped injectively onto a complement of $\text{rad}^2(Y, Z)$ in ${}_G\text{grad}(Y, Z)_{\text{End } Y}$. This completes the proof.

Now, assume that X is an indecomposable, non-injective module and that G is a radical complement in $\text{End } X$. If there are given direct complements $M(X, Y)$ of $\text{rad}^2(X, Y)$ in ${}_{\text{End } Y}\text{rad}(X, Y)_G$, then the $\sigma^*M(X, Y)$ are direct complements of $\text{rad}^2(Y, Z)$ in ${}_G\text{grad}(Y, Z)_{\text{End } Y}$, and the Auslander-Reiten sequence starting with X is of the form

$$0 \longrightarrow X \xrightarrow{(\bar{\chi}_{M(X,Y)})_Y} \oplus {}^*M(X, Y) \otimes Y \xrightarrow{(\chi_{\sigma^*M(X,Y)})_Y} Z \longrightarrow 0.$$

Denote by $c(M(X, Y))$ the canonical element in ${}^*M(X, Y) \otimes M(X, Y)$. Now $\iota : M(X, Y) \hookrightarrow \text{Hom}(X, Y)$ and $\sigma : {}^*M(X, Y) \hookrightarrow \text{Hom}(Y, Z)$, and thus we have a canonical map

$${}^*M(X, Y) \otimes M(X, Y) \longrightarrow \text{Hom}(X, Z),$$

namely $\sigma \otimes \iota$ followed by the composition map μ .

PROPOSITION 2.2. *Under the map*

$$\bigoplus_Y {}^*M(X, Y) \otimes M(X, Y) \xrightarrow{\bigoplus_Y (\sigma \otimes \iota)} \bigoplus_Y \text{Hom}(Y, Z) \otimes \text{Hom}(X, Y) \xrightarrow{(\mu)} \text{Hom}(X, Z),$$

the element $\sum_Y c(M(X, Y))$ goes to zero.

Observe that, for a fixed module X , there is only a finite number of modules Y such that $M(X, Y) \approx \text{Irr}(X, Y) \neq 0$; therefore, we may form the sum $\sum_Y c(M(X, Y))$.

Proof of Proposition 2.2. First, we are going to show that $c(M(X, Y))$ maps onto $\chi_{\sigma^*M(X, Y)} \circ \bar{\chi}_{M(X, Y)}$. Let f_1, f_2, \dots, f_d be an $\text{End } Y/\text{rad End } Y$ -basis of ${}_{\text{End } Y/\text{rad End } Y}M = M(X, Y)$, and $\phi_1, \phi_2, \dots, \phi_d$ the corresponding dual basis in ${}^*M_{\text{End } Y/\text{rad End } Y}$. Then, for $x \in X$, we have

$$\bar{\chi}_M(x) = \sum_p \phi_p \otimes f_p(x),$$

and for $\phi \in {}^*M$, $y \in Y$,

$$\chi_{\sigma}^* \star_M (\phi \otimes y) = \sigma(\phi)(y) .$$

Thus,

$$\chi_{\sigma}^* \star_M \bar{\chi}_M(x) = \chi_{\sigma}^* \star_M \left(\sum_p \phi_p \otimes f_p(x) \right) = \sum_p \sigma(\phi_p)(f_p(x)) .$$

This shows that $\chi_{\sigma}^* \star_M \bar{\chi}_M$ is equal to $\sum_p \sigma(\phi_p) f_p$, and this is the image of $\sum_p \phi_p \otimes f_p = c(M(X,Y))$ under $\mu(\sigma \otimes 1)$. As a consequence, we conclude that under the map $\bigoplus_Y^* M(X,Y) \otimes M(X,Y) \xrightarrow{\mu(\sigma \otimes 1)} \bigoplus_Y \text{Hom}(Y,Z) \otimes \text{Hom}(X,Y) \xrightarrow{(\mu)} \text{Hom}(X,Z)$, the element $\sum_Y c(M(X,Y))$ goes to $\sum_Y \chi_{\sigma}^* \star_M(X,Y) \bar{\chi}_M(X,Y)$, which is the composite of the two maps in the corresponding Auslander-Reiten sequence and thus zero. The proof is completed.

Let us point out that, in what follows, we shall not specify any longer the embedding σ of $^*M(X,Y)$ into $\text{Hom}(Y,Z)$, but shall simply consider $^*M(X,Y)$ to be a subset of $\text{Hom}(Y,Z)$.

REMARK. Let us underline the use of the two distinct tensor products $M(X,Y) \otimes ^*M(X,Y)$ and $^*M(X,Y) \otimes M(X,Y)$. Whereas the first one is used for the ordinary evaluation map

$$\chi : M(X,Y) \otimes ^*M(X,Y) \longrightarrow \text{End } Y / \text{rad End } Y$$

given by $\chi(f \otimes \phi) = f(\phi)$, it is the second one which has to be used for the composition map μ . Namely, using the above embedding $^*M(X,Y) \hookrightarrow \text{Hom}(Y,Z)$, we can consider

$$^*M(X,Y) \otimes M(X,Y) \hookrightarrow \text{Hom}(Y,Z) \otimes \text{Hom}(X,Y) \xrightarrow{\mu} \text{Hom}(X,Z) ,$$

and $\mu(\phi \otimes f) = \phi \circ f$.

3. The preprojective modules

Now, let us consider the particular case of the irreducible maps between indecomposable preprojective $R(\Omega)$ -modules. First, recall the way in which these modules can be inductively obtained from the indecomposable projective ones.

For each $i \in I$, there is an indecomposable projective $R(\Omega)$ -module $P(i)$. Indeed, denoting by e_i the primitive idempotent of $R(\Omega)$ corresponding to the identity element of the i^{th}

factor F_i in $R_O = \prod_i F_i$, $P(i) = e_i R(\Omega, \Omega)$. Note that $P(i)/\text{rad } P(i)$ is the simple $R(\Omega, \Omega)$ -module corresponding to the vertex i which defines $P(i)$ uniquely up to an isomorphism. Moreover, note that $\text{End } P(i) = F_i$, and thus it is a division ring. The irreducible maps between projective modules are always rather easy to determine. Here, for $R(\Omega, \Omega)$, there are irreducible maps from $P(j)$ to $P(i)$ if and only if $i \rightarrow j$ in Ω . In fact, ${}_i M_j$ can be easily embedded in $\text{Hom}(P(j), P(i))$ in such a way that

$${}_i M_j \oplus \text{rad}^2(P(j), P(i)) = \text{rad}(P(j), P(i))$$

as F_i - F_j -bimodules. This follows either from the explicit description of the modules $P(i)$ given in [3], or from the fact that $\oplus {}_i M_j$ is a direct complement of $\text{rad}^2 R(\Omega, \Omega)$ in $\text{rad } R(\Omega, \Omega)$. As a result, given two indecomposable projective $R(\Omega, \Omega)$ -modules P and P' , we can always choose a direct complement $M(P, P')$ of $\text{rad}^2(P, P')$ in $\text{End } P' \text{ rad}(P, P') \text{ End } P$, and we can identify these $M(P, P')$ with the given bimodules ${}_i M_j$, where $i \rightarrow j$.

Now, the indecomposable preprojective modules can be derived from the projective ones by using powers of the Coxeter functor C^- (as defined in [3]) or of the Auslander-Reiten translation $A^- = \text{Tr } D$ ("transpose of dual" of [2], and also [1]). Thus, we denote by $P(i, r)$ the module obtained from $P(i)$ by applying the r^{th} power of one of the mentioned constructions. (It is clear from the uniqueness result in [3] that $C^{-r} P(i) \approx A^{-r} P(i)$.)

LEMMA 3.1. Assume that X and Y are indecomposable modules and that there exists an irreducible map $X \rightarrow Y$. If one of the modules X, Y is preprojective, then both are. Furthermore, if $X = P(i, r)$ and $Y = P(j, s)$, then either $s = r$ and $i \leftarrow j$, or $s = r+1$ and $i \rightarrow j$.

Proof. This lemma is well-known, so let us just outline a proof. Using shifts by powers of the Coxeter functors C^+ and C^- (see [3]) or of the Auslander-Reiten translations $A = D \text{ Tr}$ and $A^- = \text{Tr } D$ (see [2] and [1]), we can assume that X is projective. If Y is not projective, then we get from the Auslander-Reiten sequence ending with Y , an irreducible map from AY to X .

Since X is projective, this map cannot be an epimorphism and thus it has to be a monomorphism. Consequently, AY is projective.

Now, in view of Proposition 2.1, we obtain by induction on the "layer" r of the indecomposable preprojective $R(\Omega)$ -modules $P(i, r)$ the following result.

PROPOSITION 3.2. a) If we choose, for any two indecomposable projective modules P and P' , a direct complement $M(P, P')$ of $\text{rad}^2(P, P')$ in ${}_{\text{End } P} \text{rad}(P, P') {}_{\text{End } P'}$, then this determines a direct complement $M(P, P')$ of $\text{rad}^2(P, P')$ in $\text{rad}(P, P')$ for any indecomposable preprojective modules P, P' .

b) If we identify, for any arrow $i \rightarrow j$ the bimodule $M(P(j), P(i))$ with ${}_{i,j}^{M_j}$, then this yields an identification of any $M(P(j, r), P(i, r))$ with ${}_{i,j}^{(2r)M_j}$ and any $M(P(i, r), P(j, r+1))$ with ${}_{i,j}^{(2r+1)M_j}$ for $i \rightarrow j$.

PROPOSITION 3.3. Every map between two indecomposable preprojective modules is a sum of composites of maps from the various $M(P, P')$.

Proof. Let Y be an indecomposable preprojective module, say $Y = P(i, r)$. Then the radical of the endomorphism ring E of

$\bigoplus_{\substack{j \in I \\ 0 \leq s \leq r}} P(j, s)$ is generated (by using the addition and multiplication) by an arbitrary complement of $\text{Rad}^2 E$ in $\text{Rad } E$. So we may choose as a complement the direct sum of $M(P(j, s), P(j', s'))$.

4. Abstract definition of the full subcategory of the preprojective modules

First, let us introduce the following notation indicating the operation of the division rings F_i and F_j : For $i \rightarrow j$, put

$${}_{i,j}^{2r}M_j = (2r)({}_{i,j}M_j) \quad \text{and} \quad {}_{j,i}^{2r+1}M_i = (2r+1)({}_{j,i}M_i).$$

Now, define the category $P(\Omega)$ as follows: The objects of $P(\Omega)$ are pairs (i, r) , $i \in I$, $r \geq 0$ with the endomorphism rings F_i . For $i \rightarrow j$,

$$M((j, r), (i, r)) = {}_{i,j}^{2r}M_j$$

and

$$M((i,r),(j,r+1)) = {}^{2r+1}_{j}M_i.$$

Denote by $F(\mathcal{A}, \Omega)$ the free category generated by these morphisms using the tensor products over F_i . Furthermore, for every (j,r) , take

$$c(j,r) = \sum_{i \rightarrow j} c({}^{2r}_{i}M_j) + \sum_{j \rightarrow k} c({}^{2r+1}_{k}M_j) \in$$

$$\oplus_{i \rightarrow j} ({}^{2r+1}_{j}M_i \otimes {}^{2r}_{i}M_j) \oplus \oplus_{j \rightarrow k} ({}^{2r+2}_{j}M_k \otimes {}^{2r+1}_{k}M_j),$$

and denote by J the category ideal generated by all elements $c(j,r)$. The category $P(\mathcal{A}, \Omega)$ is then defined as the factor category of $F(\mathcal{A}, \Omega)$ by the ideal J .

Observe that the definition of $P(\mathcal{A}, \Omega)$ requires only the knowledge of the bimodules ${}_iM_j$ for $i \rightarrow j$ (and neither the corresponding bimodules ${}_jM_i$, nor the bilinear forms ϵ_i^j and ϵ_j^i).

PROPOSITION 4.1. *The full subcategory of the preprojective modules of the category of all $T(\mathcal{A}, \Omega)$ -modules is equivalent to $P(\mathcal{A}, \Omega)$.*

Proof. Using Proposition 3.2, there is a canonical functor Γ from $F(\mathcal{A}, \Omega)$ to the subcategory of preprojective $T(\mathcal{A}, \Omega)$ -modules given by the choice of $M(P(i), P(j)) = {}_jM_i$ for projective modules $P(i), P(j)$ where $j \rightarrow i$. Also by Proposition 3.3, Γ is surjective. Moreover, according to Proposition 2.2, the elements $c(j,r)$ are mapped to zero.

Conversely, let a morphism $f : (j,r) \rightarrow (j',r')$ from $F(\mathcal{A}, \Omega)$ be mapped under Γ to zero. We are going to show that f must lie in the ideal J . This is clear if $r = r'$; for, then $f = 0$. Thus, assume that $f \neq 0$ and proceed by induction on $r' - r$. Now j and r are fixed; let $\{\dots g_p \dots\}$ be the union of bases of all vector spaces ${}_i({}^{2r}_{i}M_j)$ for all $i \rightarrow j$ and ${}_k({}^{2r+1}_{k}M_j)$ for all k with $j \rightarrow k$, and let $\{\dots g'_p \dots\}$ be the union of the corresponding dual bases of $({}^{2r+1}_{j}M_i)_{F_i}$ and $({}^{2r+2}_{j}M_k)_{F_k}$.

Thus, $c(j,r) = \sum_p g'_p \otimes g_p$. Now, $f = \sum_p h_p \otimes g_p$, where h_p is a morphism of $F(\mathcal{M}, \Omega)$ either from (i,r) or $(k,r+1)$ to (j',r') . Since there is an Auslander-Reiten sequence

$$0 \longrightarrow P(j,r) \xrightarrow{(\Gamma(g_p))_p} Q \xrightarrow{(\Gamma(g'_p))_p} P(j,r+1) \longrightarrow 0$$

and since

$$0 = \Gamma(f) = \sum_p \Gamma(h_p) \Gamma(g_p),$$

we can factor $(\Gamma(h_p))_p : Q \rightarrow P(j',r')$ through $(\Gamma(g'_p))_p$. Hence, there is a homomorphism $\tilde{u} : P(j,r+1) \rightarrow P(j',r')$ such that

$$\Gamma(h_p) = \tilde{u} \Gamma(g'_p).$$

And, since Γ is surjective, we can find $u : (j,r+1) \rightarrow (j',r')$ in $F(\mathcal{M}, \Omega)$ such that $\Gamma(u) = \tilde{u}$. Obviously, the elements $h_p - u \otimes g'_p$ lie in the kernel of Γ , and therefore, by induction, they belong to J . Consequently,

$$f = \sum_p h_p \otimes g_p = \sum_p (h_p - u \otimes g'_p) \otimes g_p + \sum_p u \otimes g'_p \otimes g_p$$

also belongs to J ; for, $\sum_p u \otimes g'_p \otimes g_p = u \otimes c(j,r)$.

5. Proof of the theorem

The proof of the theorem consists in identifying the additive structure of $\Pi(\mathcal{M})$ with a factor of a subcategory of $F(\mathcal{M}, \Omega)$. Indeed, we may consider both $F(\mathcal{M}, \Omega)$ and $P(\mathcal{M}, \Omega)$ defined in section 4 as abelian groups forming the direct sum of all $\text{Hom}((i,r), (j,s))$. Denote by $\Phi(\mathcal{M}, \Omega)$ and $\Pi(\mathcal{M}, \Omega)$ the respective subgroups of all $\text{Hom}((i,0), (j,s))$. Then, both $\Phi(\mathcal{M}, \Omega)$ and $\Pi(\mathcal{M}, \Omega)$ contain a subring $R = \bigoplus_{i,j} \text{Hom}((i,0), (j,0))$ which is obviously isomorphic to $R(\mathcal{M}, \Omega)$. Furthermore, under the composition in $\Pi(\mathcal{M}, \Omega)$, $\Pi(\mathcal{M}, \Omega)$ is a right $R(\mathcal{M}, \Omega)$ -module; for, if $f : (i,0) \rightarrow (j,s)$ and $a : (k,0) \rightarrow (i,0)$ from R , then $fa : (k,0) \rightarrow (j,s)$ in $\Pi(\mathcal{M}, \Omega)$.

PROPOSITION 5.1. $\Pi(\mathcal{M}, \Omega)_{R(\mathcal{M}, \Omega)}$ is isomorphic to the direct sum of all \bigvee preprojective $R(\mathcal{M}, \Omega)$ -modules (each occurring with multiplicity one).

\bigvee = indecomposable

Proof. Using the notation of section 3, the indecomposable preprojective R-modules are $P(j,s)$, $j \in I$, $s \geq 0$. In particular, $P(j,0)$ are the indecomposable projective R-modules and thus

$$\begin{aligned} R_R &= \bigoplus_{i \in I} P(i,0). \text{ For every R-module } X_R, \\ X_R &\approx \text{Hom}(R_R, X_R) = \text{Hom}\left(R, \left[\bigoplus_i P(i,0)\right], X_R\right) = \\ &= \left[\text{Hom}\left(\bigoplus_i P(i,0), X_R\right)\right]_R = \left[\bigoplus_i \text{Hom}(P(i,0), X_R)\right]_R. \end{aligned}$$

Hence,

$$P(j,s) = \left[\bigoplus_i \text{Hom}(P(i,0), P(j,s))\right]_R$$

and thus under the identification of $P(j,s)$ with (j,s) and $\text{Hom}(P(i,0), P(j,s))$ with the maps in $\Pi(\mathcal{M}, \Omega)$, we get the statement.

Now, define the map $\Delta : T(\mathcal{M}) \rightarrow F(\mathcal{M}, \Omega)$ as follows. First, the morphisms in $F(\mathcal{M}, \Omega)$ can be described in the following way: For an (unoriented path) $w = i_{n+1} - i_n - \dots - i_2 - i_1$ of \mathcal{M} , call the number of arrows $i_{t+1} \leftarrow i_t$, $1 \leq t \leq n$, in Ω the layer $\lambda(w)$ of w . Then, the morphisms in $F(\mathcal{M}, \Omega)$ are the elements of the tensor products

$$i_{n+1}^{r_{n+1}} M_{i_n} \otimes \dots \otimes i_3^{r_3} M_{i_2} \otimes i_2^{r_2} M_{i_1},$$

$$\text{where } r_t = 2\lambda(i_t - i_{t-1} - \dots - i_2 - i_1) + \begin{cases} 0 & \text{if } i_{t+1} \rightarrow i_t \\ 1 & \text{if } i_{t+1} \leftarrow i_t \end{cases}.$$

Now, the map Δ is defined by

$$i_{n+1}^{M_{i_n}} \otimes \dots \otimes i_3^{M_{i_2}} \otimes i_2^{M_{i_1}} \xrightarrow{i_{n+1}^{r_{n+1}} \otimes \dots \otimes i_3^{r_3} \otimes i_2^{r_2}} i_{n+1}^{r_{n+1}} M_{i_n} \otimes \dots \otimes i_3^{r_3} M_{i_2} \otimes i_2^{r_2} M_{i_1},$$

where r_η are the maps of Lemma 1.2 for $M = i^{M_j}$ and $N = j^{M_i}$.

From the definition of $\Phi(\mathcal{M}, \Omega)$, it is clear that $\Phi(\mathcal{M}, \Omega)$ is just the image of $T(\mathcal{M})$ under Δ . Also, Δ is obviously $R(\mathcal{M}, \Omega)$ -linear.

LEMMA 5.2. $\Delta(\langle c \rangle) = J \cap \Phi(\mathcal{M}, \Omega)$.

Proof. By definition, $c = \sum_j (\sum_i c_j^i) = \sum_j c(j)$; note that $c(j) = e_j c e_j$, where e_j is the idempotent of $T(\mathcal{M})$ corresponding

to the identity of F_j ; thus $\langle c \rangle$ is the ideal generated by all $c(j)$'s. Hence, the statement follows from Lemma 1.2 taking into account that, by definition,

$$\Delta(1 \otimes 1 \otimes \dots \otimes c(j) \otimes \dots \otimes 1) = 1 \otimes 1 \otimes \dots \otimes c({}^T M) \otimes \dots \otimes 1.$$

Now, from Lemma 5.2, it follows that Δ defines an isomorphism of $\Pi(\mathcal{M}) = T(\mathcal{M})/\langle c \rangle$ onto $\Pi(\mathcal{M}, \Omega) = \Phi(\mathcal{M}, \Omega)/J \cap \Phi(\mathcal{M}, \Omega)$. This completes the proof of the theorem.

The corollaries follow from the results in [2].

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